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ORIGINAL PAPER

A flow rule incorporating the fabric and non-coaxiality in granular materials

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Abstract Deficiencies of constitutive models in prediction of dilatancy are often attributed to simplifications associated with flow rules such as assumptions of isotropy and coaxiality. It is thus proposed here to develop a comprehensive flow rule for granular materials by including the effect of fabric and without the assumption of coaxiality. A second-order tensor is introduced as a fabric for the distribution of contact normals and contact forces. By using the energy principle in micro-mechanical scale and a suitable dissipation mechanism in granular materials, a stress-dilatancy relation is obtained. Fabric plays a "bridge-like" role in the dilatancy and non-coaxiality. Non-coaxialities between stress-strain-fabric are attributed to the non-coaxiality between stress-fabric and strain-fabric. In this formulation the constants for modeling fabric depend on non-coaxiality of the system rather than the history that determines such a state. Ability of this stress-fabric-dilatancy for modeling the non-coaxiality shows that this relation can predict the behavior of granular materials in the presence of the rotation of principal stress axes.

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1 Introduction

Prediction of dilatancy of granular materials still poses a challenge issue in constitutive modeling. Stress-dilatancy theories have been established originally based on energy principles [3,11–14,31,43,48,56,57]. Determination of the stress-strain and failure behavior of the cohesionless soils are quite complex due to their granular structure [55]. By defining a homogenization technique from local phenomena to the global behavior, this complex behavior can be clarified. Some of the stress-dilatancy relations have been developed based on the constraint imposed on the micromechanical behavior of grains [9,24,28,53]. However, the energy principle has been used as dissipation function in different micro- and macro scales and thus various dilatancy relations have been proposed.

Two issues of importance in modeling of granular materials are associated with the internal microstructure (or fabric) and non-coaxialities between stress, strain and fabric [6,21,56–58]. Micromechanical analysis has been shown to be a viable approach to develop the effect of fabric and its evolution [58]. In the realm of micromechanical formulations there are two main approaches, one is fabric embedded plasticity models and the other is purely micromechanical models [58]. In the former, a fabric tensor is assumed to characterize the mechanical properties of granular materials. Hence, in macroscopic plasticity models dilatancy is included through the flow rule. The microstructural dependency of dilatancy is derived from micromechanical considerations, but with different rules hypothesized. By assuming different mechanisms for the evolution of fabric or contact arrangement with shearing deformation, different flow rules will be obtained [6,21,56].

For purely micromechanical models the approach provides a physical insight to treat granular particles at the local force (static) versus displacement (kinematic). Whereas in the approach that utilizes homogenization techniques, both micro-level kinematic and static variables are be linked to their macro-level counterparts, i.e., stress, strain and microstructure arrangement (or fabric) of the granular assembly. In order to characterize the microscopic parameters, some phenomenological rules must be hypothesized. Chang [4] and Nemat-Nasser [29] used this approach in their formulations. In order to combine the non-coaxiality and fabric anisotropy Nemat-Nasser [29] and Nemat-Nasser and Zhang [30] included fabric in the dilatancy equation. In micro-level analysis, they attributed the shearing resistance to the Coulomb type friction and the fabric anisotropy.

In the literature of constitutive modeling the effect of fabric and non-coaxiality between stress and fabric accounted for into the calculation of dilatancy by considering it as a constraint imposed by internal grain geometry on macro-level deformation [6,9,21,22,24,56,57,62]. By definition of microstructural arrangement of particles, stress and strain must be related through the particles contacts. Thus, the non-coaxiality between the stress and the strain has to be established via the fabric anisotropy. In other words, non-coaxiality must be defined between stress and fabric and strain and fabric.

In this paper a physically based dilatancy relation is proposed without the hypothesis of coaxiality and by linking discrete micromechanics and continuum mechanics. For this purpose, initially a micro-analysis is carried out on the contact normal and contact normal forces distribution function by accounting the effect of fabric on the mean behavior of particles through a simple local contact law. Then the laws of thermodynamics are utilized to derive the internal supplied and dissipated energies. In contrast to previous works where the non-coaxiality has been attributed to the stress and strain increment (e.g., [10,11,40,61]) or to the stress and fabric [21,41,56,57], the microstructural parameters of the proposed flow are explicitly related to the deviatoric load with respect to non-coaxiality between stress-strain-fabric. Furthermore, the constants used in the description of the fabric and its evolution depend only on the non-coaxiality between stress and fabric irrespective of bedding angles or confining pressures. Whereas in other models the constants are based on the history of deformation. Non-coaxialities between stress, strain and fabric will be discussed, verified and compared with other flow rules in geomechanics.

2 Direction of contact normals and contact force distributions

Rothenburg and Bathurst [46,47] proposed equations for contact normal and contact force distributions and attributed the strength of granular materials to these parameters. Particles are packed in two cases: isotropic and anisotropic. In the isotropic case particles are in regular form and equal contacts. The contact normals are distributed uniformly in the whole assembly. After shearing the contact normals and consequently the contact normal forces change in a continuous manner [24,35,46]. Shearing deformations lead to anisotropy in the contact normal forces because the number of load carrying contacts and their distributions change. In the shearing process, anisotropy is developed with the disruption and generation of new contacts [24,25,37,38]. Kruyt [16] quantify the fabric evolution by considering three mechanisms: contact generation, contact disruption, and contact reorientation. It was found that the contact disruption is the dominant mechanism [16]. The dissipation of energy inside the media is attributed to these mechanisms [15,49]. The changes of these mechanisms and their dissipations should be included in the derivation of the energy method which is used to obtain stress-fabric-dilatancy equation.

The energy method is a basic approach to formulate dilatancy in granular geomaterials (e.g., [5,28,44,48]). In the energy methods two functions have to be defined, i.e. applied (external) energy and dissipated energy. Wan and Guo [56,57] used macro-level approach to obtain the applied energy, but micro-level analysis to obtain the dissipation function. Nemat-Nasser [28,29] used a micro-level analysis for the applied loads and considered the dissipation function as proposed by Roscoe [45]. However, it is reasonable to obtain functions for both the applied and the dissipated energy mechanism in a same level.

3 Total work input

In order to determine the total work input in microscopic level the mean field theory is employed. Applied or external energy may be related to the microstructure of the granular materials. Based on the relation proposed by Emeiault and Cambou [7], the applied energy function may be defined as follows:

$$\sigma_{ij} \cdot \varepsilon_{ij} = \oint E(\theta) f_i d_j d\Omega \tag{1}$$

where σ_{ij} is the Cauchy stress, ε_{ij} its counterpart strain, f_i is the internal force, d_j is relative displacements of contact

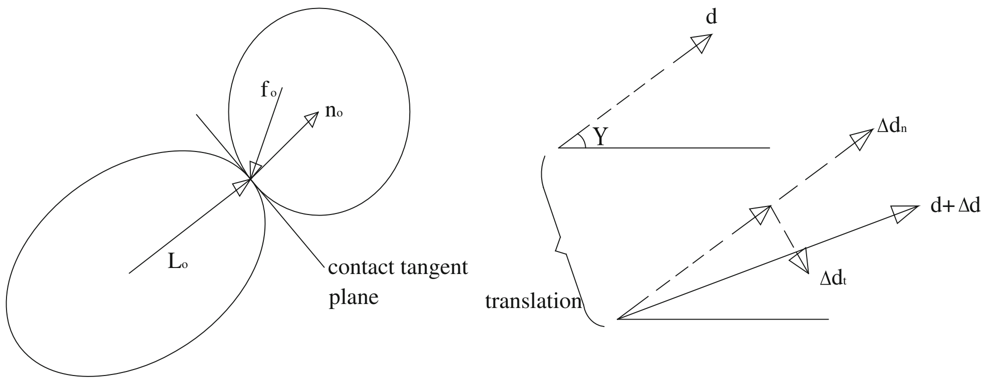

Fig. 1 Contact normal and contact forces in the kinematics of uniform deformation

points and $E(\theta)$ is the distribution function of the contacts. In the above equation the balance between applied or macroscopic work and the integral of the microscopic work overall contact direction is established. To develop Eq. (1) a function is necessary to describe the distribution of the contact normals, the contact forces and their counterpart displacements in the micro-level. However, these variables are very complex [7] and for this reason as a first approximation it is proposed to consider the mean values of the variables for each contact orientation. By using the Fourier series and the distinct element method (DEM), Rothenburg and Bathurst [46,47] showed that for the circular and elliptical granular assemblies can be defined by:

$$E(\theta) = (1/2\pi)(1 + \alpha \cos 2(\theta - \theta_f)) \tag{2}$$

where α is the magnitude of the anisotropy and θ_f is the major direction of the contact normals with respect to the major principal stress. In order to analyze the micro-level origin of the shear strength a statistical description of the force transmission and its distribution are needed. Rothenburg and Bathurst [47] and Radjai and Azema [42] proposed harmonic approximations for the internal forces density distribution functions such as:

$$f_n(\theta) = \langle f_\circ \rangle \left[1 + \alpha_n \cos 2(\theta - \theta_n)\right] \tag{3}$$

$$f_t(\theta) = \langle f_\circ \rangle \alpha_t \sin 2(\theta - \theta_t) \tag{4}$$

$f_\circ$ is the mean radial force, α_n and α_t are the anisotropies of the radial and tangential forces respectively, θ_n and θ_t are the directions of the radial and tangential forces with respect to the principal stress axis. Deformation in the microscopic level can be related to the macroscopic deformations via the following equations [47]:

$$(d_n/\delta) = (\dot{\varepsilon}_v + \dot{\varepsilon}_q \cos 2(\theta - \theta_\varepsilon))/2 \tag{5}$$

$$(d_t/\delta) = -(\dot{\varepsilon}_q \sin 2(\theta - \theta_\varepsilon))/2 \tag{6}$$

where $\dot{\varepsilon}_v$ and $\dot{\varepsilon}_q$ are the volumetric and deviatoric strains, respectively, θ_ε is the direction of the major principal strain with respect to the horizontal axis; d_n and d_t are the deformations in the radial and tangential direction, respectively, as shown in Fig. 1 and δ stands for the increment of deformation. The balanced applied energy in the micro-level is obtained by substituting Eqs. (2)–(6) in Eq. (1) such that:

$$\begin{aligned}
\oint E(\theta) f_i d_j d\Omega &= \int_0^{2\pi} [(f_n(\theta).\dot{d}_n) + (f_t(\theta).\dot{d}_t)] E(\theta) d_\theta \\
&= \int_0^{2\pi} f_n(\theta).d_n E(\theta) d\theta + \int_0^{2\pi} f_t(\theta).d_t E(\theta) d\theta \\
&= \int_0^{2\pi} f_\circ(1 + \alpha_n \cos 2(\theta - \theta_n))(\dot{\varepsilon}_v + \dot{\varepsilon}_q \cos 2(\theta \\
&\quad - \theta_\varepsilon))(1/2\pi)(1 + \alpha \cos 2(\theta - \theta_f)) d\theta \\
&\quad + \int_0^{2\pi} f_\circ(\alpha_t \sin 2(\theta - \theta_\varepsilon))(\dot{\varepsilon}_q \sin 2(\theta \\
&\quad - \theta_\varepsilon))(1/2\pi)(1 + \alpha \cos 2(\theta - \theta_f)) d\theta
\end{aligned} \tag{7}$$

After integrating and mathematical manipulation the following equation is obtained:

$$\begin{aligned}
\oint f_i d_j = \left(\frac{f_\circ}{4}\right) &[(1 + \alpha\alpha_n \cos 2(\theta_n - \theta_f))\dot{\varepsilon}_v \\
&+ (1/2)((\alpha \cos 2(\theta_f - \theta_\varepsilon) + \alpha_n \cos 2(\theta_n - \theta_\varepsilon) \\
&+ \alpha_t \cos 2(\theta_f - \theta_\varepsilon))\dot{\varepsilon}_q]
\end{aligned} \tag{8}$$

Neglecting the effect of $\alpha\alpha_n$, and for the non-coaxiality between the major principal direction of the stress tensor and the major principal direction of the different components of the fabric tensors, and also by neglecting the cross products among the anisotropies Radjai et al. [41] proposed the following equations:

$$\begin{aligned}
(q/p_\circ) \cong (1/2)[&\alpha \cos 2(\theta_\sigma - \theta_f) \\
&+ \alpha_n \cos 2(\theta_\sigma - \theta_n) + \alpha_t \cos 2(\theta_\sigma - \theta_t)]
\end{aligned} \tag{9}$$

where θ_σ is the direction of the principal stress tensor. Combining Eqs. (8) and (9) to obtain the applied energy in the presence of the non-coaxialities between stress-strain-fabric, thus

$$\oint f_i d_j d\Omega = (f_\circ/4)\{[(1+\alpha\alpha_n \cos 2(\theta_n - \theta_f))]\dot{\varepsilon}_v + (1/2)[(\alpha \cos 2(\theta_\sigma - \theta_f)\cos 2(\theta_f - \theta_\varepsilon)) + \alpha_n \cos 2(\theta_\sigma - \theta_n)\cos 2(\theta_n - \theta_\sigma) + \alpha_t \cos 2(\theta_\sigma - \theta_t)\cos 2(\theta_f - \theta_\varepsilon)]\dot{\varepsilon}_q\} \quad (10)$$

In the above equation non-coaxialities in the total work input are established between stress-fabric and strain-fabric.

4 Dissipation function

External loads and applied energy are dissipated by the friction between particles and the displacement and/or the rotation of them. Different mechanisms to dissipate energy have been taken into account by different researchers. In macro-level, Roscoe et al. [43], Roscoe [45] and Nemat-Nasser [29] attributed dissipation to the friction sliding of the granular particles. Roscoe et al. [43] showed that the rate of dissipation in the failure is $q\dot{\varepsilon}_q$, where q is the deviatoric stress. They also found that the pre-failure rate of dissipation is equal to $q\dot{\varepsilon}_q$, and at failure $q = Mp_\circ$, hence the rate of dissipation, $\dot{\psi}$ is equal to:

$$\dot{\psi} = p_\circ M\dot{\varepsilon}_q \quad (11)$$

Nemat-Nasser [29] and Nemat-Nasser and Zhu [30] used this rate of dissipation in their formulations. Nova and Muir Wood [33] and later Jefferries [14] found that the rate of dissipation has another term, i.e.:

$$\dot{\psi} = p_\circ M\dot{\varepsilon}_q + p_\circ N\dot{\varepsilon}_v \quad (12)$$

where N is an amount of the volumetric work associated with the stress-dilatancy. Nova [32] considered N as a density-independent material property.

For the first order rate of dissipation of energy and by considering pure stress rotation Vardoulakis and Georgopoulos [54] used the following equation for the non-coaxiality case between stress and strain:

$$\dot{\psi} = q\dot{\varepsilon}_q \cos\delta + p_\circ\dot{\varepsilon}_v \quad (13)$$

where $\cos\delta$ is similar to the non-coaxiality parameter that was proposed by Gutierrez and Ishihara [12]. After some simplification due to the coaxial and non-coaxial part of Eq. (13), they proposed the following equation:

$$\dot{\psi} = p\dot{\varepsilon}_q f_c \quad (14)$$

where f_c is a constant parameter which depends on the type of the sand, (e.g., $f_c \simeq 0.48$ for the Toyoura sand) [12]. The effect of fabric was not directly included in the dissipation mechanism by the above constitutive models.

The effect of fabric in the dissipation mechanism must be included in the micro-level via the number of disruption and generation of contacts and contact normal surfaces. Kruyt and Rothenburg [15] suggested that the different dissipation mechanism in the microscopic level must be investigated and then linked it to the macroscopic dissipative characteristics. For this reason, confining pressure and fabric must be included in the dissipation mechanism. Wolf et al. [60] introduced two types of irreversible interaction: (1) energy dissipation due to incomplete normal restitution (which means relative velocities after and before the collision of particles) in head on collision, (2) energy dissipation during sliding of the granular materials based on Coulomb friction law.

Kruyt and Rothenburg [15] by using DEM added two parameters to account for numerical dissipation. Wang and Zhu [59] attributed dissipation to two phenomena, i.e. the fragile dissipation and the rheological dissipation. Wan and Guo [55,56] were first to explicitly incorporate the fabric and its evolution in the constitutive modeling of granular materials. By assuming that energy dissipation is purely frictional, they included only the tangential terms in their formulations. They showed that the rate of energy dissipation per contact point, $\dot{\psi}$ is:

$$\dot{\psi} = N\dot{d} + \dot{N}d \quad (15)$$

where N is the number of contacts and d is the dissipation per contact. The dot over d and N imply that the derivative and changes of the parameters with shearing. In the granular assembly, the velocity jump has only a shear component [39], for this reason the rate of energy dissipation per contact based on the Coulomb friction equals to:

$$\dot{d} = \langle f_t \rangle \cdot \langle \Delta\dot{u}_t \rangle \quad (16)$$

where $\langle\cdot\rangle$ stands for the volume average taken over volume mass. For the first part of the dissipation mechanism, the rate of frictional energy is:

$$N\dot{d} = N \cdot \langle f_t \rangle \cdot \langle \Delta\dot{u}_t \rangle \quad (17)$$

Sliding in the granular materials follow the Coulomb's law:

$$\langle f_t \rangle = \mu(x) \cdot \langle f_n \rangle \quad (18)$$

where $\mu(x)$ is the kinematic friction coefficient. Radial force is a function of the confining pressure [39]

$$\langle f_n \rangle = \phi(x) \cdot p_\circ \quad (19)$$

where $p_\circ$ is the confining pressure. By substituting Eq. (18) and (19) in Eq. (17), the following is obtained:

$$N\dot{d} = N \cdot \mu(x) \cdot \phi(x) \cdot p_\circ \cdot \langle \Delta\dot{u}_t \rangle \quad (20)$$

where $\phi(x)$ is a scalar function which strongly depends on the microstructures (fabric, density,...) of the granular medium and the loading conditions. Tangential deformation in the micro-level may be related to the deviatoric (shear) deformation in the macro-level. Applying this assumption to the first part of the dissipation mechanism:

$$N\dot{d} = N \cdot \mu(x) \cdot \phi(x) \cdot p_\circ \cdot \dot{\varepsilon}_q \tag{21}$$

Since the total number of contacts per unit volume does not change at critical state [22,62] and also the real stress approaches a constant value as reflected by the inter-particle friction, it must be related to the stress ratio at critical state. The amount of $N\mu(x)\phi(x)$ is a function of fabric, density and frictional sliding (frictional failure) of the medium (e.g., [56,57]), thus

$$N \cdot \mu(x) \cdot \phi(x) = f(\psi, M, F_{ij}) \tag{22}$$

where M is the stress ratio at critical state. By assuming that soil is a distortional material, the current size of the yield surface depends on the plastic distortional strain ε_q^p [27]. Moreover, the effect of fabric is more pronounced in the first steps of shearing with increasing of distortional shear deformation [34,36]. Combining the relations proposed by Li and Dafalias [20,21] and Wan and Guo [55,56] results in:

$$\begin{aligned} &N \cdot \mu(x) \cdot \phi(x) \\ &\quad = \frac{[(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)]X+\varepsilon_q^p}{c+\varepsilon_q^p} \\ &\qquad \times M\exp(n_d\psi) \end{aligned} \tag{23}$$

By substituting Eq. (23) into Eq. (21) we obtain

$$\begin{aligned} &N\dot{d} \\ &\quad = \frac{[(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)]X+\varepsilon_q^p}{c+\varepsilon_q^p} \\ &\qquad \times M\exp(n_d\psi)p_\circ\dot{\varepsilon}_q \end{aligned} \tag{24}$$

where X is a parameter that depends on the confining pressure, ψ is a state parameter, and c is a material constant. This parameter is directly related to the confining pressure; hence the term $\exp(n_d\psi)$ can be neglected.

The second term in the dissipation function [Eq. (15)] is directly due to the changes of fabric. $\dot{N}$ is the change of contacts and one of the main source of anisotropy in the granular mass which evolves with the deformation. Lanier and Calvetti [18] showed that:

$$\dot{N}(\theta,\Delta\theta) = N(\theta,\Delta\theta)b'n_in_j\Delta\varepsilon_{ij} \tag{25}$$

where $\dot{N}(\theta,\Delta\theta)$ is the changes of the distribution of the contact normals, b' imply the hardening function of the materials, $\Delta\varepsilon_{ij}$ is the increment of strain, and n_i and n_j are the directions of the contact normals with respect to the orthogonal coordination, X and Y, respectively. Because of the distortional hardening characteristics of the granular geomaterials, b' may be a function of the plastic shear strain. Disruption and generation of the contacts is the result of shearing and cause the anisotropy of the contact normals and contact forces. Most of the contacts are lost in the tensile direction and generated in the compressive direction [8,25].

Rolling and sliding are the main factors in the deformation and the dissipation of energies in the granular materials [2,37]. Although sliding contacts are more than rolling contacts, sliding contacts dissipate much more energy than rolling contacts, and occurs in the weak network forces [2,42]. Therefore, sliding of contacts is important factor in the dissipation of energy. Dissipated energy due to the changes of the contacts may be developed in the following manner (the effect of opening/closing is not included). Increment of the strain in Eq. (25) can be shown by the increments of the volumetric and deviatoric strains, as follows [18]:

$$\begin{aligned} \dot{N}(\theta,\Delta\theta) &= N(\theta,\Delta\theta)b'n_in_j\Delta\varepsilon_{ij} \\ &= b'n_in_jN(\theta,\Delta\theta)(\dot{\varepsilon}_v+\dot{\varepsilon}_q\cos 2(\theta-\theta_\varepsilon)) \end{aligned} \tag{26}$$

By substituting Eq. (26) into Eq. (25) we will have:

$$\begin{aligned} \dot{N}d &= \int_0^{2\pi} b'n_in_jN(\theta,\Delta\theta)(\dot{\varepsilon}_v+\dot{\varepsilon}_q\cos 2(\theta-\theta_\varepsilon)) \\ &\quad (f_\circ(1+\alpha_n\cos 2(\theta-\theta_\varepsilon)))(f_\circ\alpha_t\sin 2(\theta-\theta_t))d\theta \end{aligned} \tag{27}$$

After integrating, the following equation is obtained:

$$\dot{N}d = F(F_{ij},\varepsilon_q^p)f_\circ\dot{\varepsilon}_v[1+\alpha\alpha_n\cos 2(\theta_n-\theta_f)] \tag{28}$$

The dissipation function is finally obtained by combining Eqs. (24) and (28)

$$\begin{aligned} \dot{\bar{\psi}} &= \frac{(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)X+\varepsilon_q^p}{C+\varepsilon_q^p} \\ &\quad \times M\exp(n^d\psi)p_\circ\dot{\varepsilon}_q \\ &\quad + F(F_{ij},\varepsilon_q^p)f_\circ[1+\alpha\alpha_n\cos 2(\theta_n-\theta_f)]\dot{\varepsilon}_v \end{aligned} \tag{29}$$

As suggested by Kruyt and Rothenburg [15], the above equation links the dissipative mechanism in the micro-level to the macro-level dissipative characteristics. For granular geomaterials such as sand, the elastic energy is negligible and consequently all the applied energy by the internal force is dissipated. Here too it is assumed that the effect of elastic stored energy is negligible and all the work done by the internal forces is dissipated (e.g., [12,14,15,53]), thus

$$\oint E(\theta)f_id_jd\Omega = \dot{\bar{\psi}} \tag{30}$$

Hence,

$$(f_\circ/4)\{[(1+\alpha\alpha_n\cos 2(\theta_n-\theta_f))]\dot{\varepsilon}_v$$

$$+(1/2)[(\alpha\cos 2(\theta_\sigma-\theta_f)\cos 2(\theta_f-\theta_\varepsilon))$$
$$+\alpha_n\cos 2(\theta_\sigma-\theta_n)\cos 2(\theta_n-\theta_\sigma)$$
$$+\alpha_t\cos 2(\theta_\sigma-\theta_t)\cos 2(\theta_f-\theta_\varepsilon)]\dot{\varepsilon}_q\}$$
$$=\frac{(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)X+\varepsilon_q^p}{C+\varepsilon_q^p}$$
$$\times M\exp(n^d\psi)p_\circ\dot{\varepsilon}_q$$
$$+F(F_{ij},\varepsilon_q^p)f_\circ[1+\alpha\alpha_n\cos 2(\theta_n-\theta_f)]\dot{\varepsilon}_v \quad (31)$$

Dividing both sides of the above equation by $p_\circ\dot{\varepsilon}_q$ or its equivalent in the micro-level $p_\circ=f_\circ(1+\alpha\alpha_n\cos 2(\theta_n-\theta_f))$ [47] and rearrangement gives:

$$\frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_q}=\frac{\frac{(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)X+\varepsilon_q^p}{C+\varepsilon_q^p}M\exp(n^d\psi)-nc_{\sigma-f}\,nc_{\varepsilon-f}\,\eta}{1-F(F_{ij},\varepsilon_q^p)} \quad (32)$$

in the above equation:

$$nc_{\sigma-f}nc_{\varepsilon-f}\eta$$
$$=(1/2)\frac{(\alpha\cos 2(\theta_\sigma-\theta_f)\cos 2(\theta_\varepsilon-\theta_f)+\alpha_n\cos 2(\theta_\sigma-\theta_f)\cos 2(\theta_\varepsilon-\theta_f)+\alpha_t\cos 2(\theta_\sigma-\theta_f)\cos 2(\theta_\varepsilon-\theta_f))}{1+\alpha\alpha_n\cos 2(\theta_n-\theta_f)} \quad (33)$$

Equation (32) can be shown as:

$$\bar{d}=\frac{1}{1-F(F_{ij},\varepsilon_q^p)}(G'M^d-nc_{\sigma-f}nc_{\varepsilon-f}\eta) \quad (34)$$

where $\bar{d}$ is dilatancy, $nc_{\sigma-f}$ is the non-coaxiality between stress and fabric, $nc_{\varepsilon-f}$ is the non-coaxiality between strain and fabric, $F(F_{ij},\varepsilon_q^p)$ is the state of fabric evolution, and G' is a function expressed by the following equation:

$$G'=\frac{(1+(1/2)\alpha\cos 2(\theta_\sigma-\theta_f))\cos 2(\beta_i-\beta_\circ)X+\varepsilon_q^p}{C+\varepsilon_q^p} \quad (35)$$

Equation (33) shows that the non-coaxiality between the major principal direction of the stress (θ_σ) and the major principal direction of the strain (θ_ε) are related via the fabric θ_f. In other words, fabric acts like a "bridge" between these two separate parts. The F function in the denominator of Eq. (32), as mentioned before, is a function of fabric which can be shown by the magnitude of anisotropy α and the major direction of fabric θ_f. Baker and Desai [1] proposed an equation for the intensity of anisotropy, T as:

$$T=\frac{3(A\varepsilon_q^p)}{1+(A\varepsilon_q^p)} \quad (36)$$

They described the importance of A and its effect either as a constant or as a function of the plastic shear strain. Here, by considering Eqs. (36) and (2) and that the yield surface and the plastic potential intersect in the yielding point, the following equation is suggested:

$$F(F_{ij},\varepsilon_q^p)=2^{(\varepsilon_q/\varepsilon_{q\max})}(0.5)\left(\frac{2-(0.5\alpha\cos 2(\theta_\sigma-\theta_f))}{4+(0.5\alpha\cos 2(\theta_\sigma-\theta_f))}\right) \quad (37)$$

where $\varepsilon_{q\max}$ is the shear strain corresponding to the maximum shear stress. Since the magnitude of anisotropy α and the non-coaxiality approach constant values [16,17], Eq. (32) tends to a constant value in the critical state [22,62]. Evolution of the anisotropic parameters such as α and θ_f have an important effect on the dilatancy regime. Direct calculation of the fabric parameters is presented in the following section.

5 Fabric evolution

The parameters α and θ_f show the status of the fabric and its evolution. These parameters have a great influence on the behavior of the dilatancy equation. Taha and Shaverdi [52] proposed an equation which can predict the magnitude of α and θ_f in the presence of the non-coaxiality. This equation is obtained from the micro-level analysis. To calculate the α parameter, the magnitude of the shear to normal stress ratio on the spatial mobilized plane (SMP) must be determined. In the triaxial case, for example, τ/p may be obtained from the following equation [26]:

$$\tau/p=\sqrt{\sigma_1/\sigma_3}-\sqrt{\sigma_3/\sigma_1} \quad (38)$$

The parameters α and θ_f may be obtained from the following equations in the presence of non-coaxiality [52]:

$$\alpha=\frac{(\tau/p)\cos\phi_{\mu mob}-\sin\phi_{\mu mob}}{\sin(2\theta_f+\phi_{\mu mob})-((\tau/p)\cos(2\theta_f+\phi_{\mu mob}))} \quad (39)$$

$$\dot{\theta}_f = \dot{\theta}_\sigma + (1/2) \cdot d\eta \cdot (\theta_\sigma - \theta_f) \tag{40}$$

where the dot over θ shows the variation. The most important parameter in the above equation is the inter-particle mobilized friction angle, $\phi_{\mu mob}$. This parameter is obtained from the following equation:

$$\tan^{-1}\left(\frac{\tau}{p}\right) = \frac{\theta_\sigma - \theta_f}{z} + \lambda\left(\frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_q}\right) + \phi_{\mu mob} \tag{41}$$

where z and λ are material constants. These constants depend on the non-coaxiality between stress and fabric. z and λ are the constants used for the mobilization of the inter-paricle friction angle in the presence of anisotropy. Kuhn [17] and Shaverdi et al. [50] showed that the variation of α with the shear strain is similar to the variation of shear to normal stress ratio with shear strain.

6 Discussion

The non-coaxialities between stress-fabric and strain-fabric are established in Eq. (32). In this equation fabric acts as a bridge-like role to link the stress tensor to the strain tensor. Jefferies [14] dilatancy formulation can be obtained by assuming the function $F(F_{ij}, \varepsilon_q^p)$ in the denominator to be constant (N'), and by neglecting the effect of fabric and state parameter on the critical stress ratio M and also neglecting the non-coaxialities:

$$\bar{d} = \frac{(M - \eta)}{1 - N'} \tag{42}$$

If the fabric and non-coaxiality between stress and fabric are only taken into account, the Dafalias and Manzari [6] flow rule will be obtained:

$$\bar{d} = A_d(M^d - \eta) \tag{43}$$

If the fabric anisotropy is not taken into account and the coaxiality exists between strain and fabric, then Eq. (32) becomes equal to the Gutierrez et al. [10] flow rule:

$$\bar{d} = M - n_{\sigma-\varepsilon}\eta \tag{44}$$

Following the above discussion, it may be noted that the presented flow rule [Eq. (32)] is inclusive of all previously proposed equations and thus more comprehensive.

7 Verification with experimental tests

The initial density, fabric, and non-coaxiality are the main factors that affect the dilatancy in granular geomaterials. The effect of initial density is included in the constitutive equations and has been verified with the experimental tests, especially for the dilatancy equation by Manzari and Dafalias [23], Li and Dafalias [21]. Non-coaxiality and its effect, has also been incorporated and verified with experimental tests by Gutierrez and Ishihara [11,12], Yu [61], and Lashkari and Latifi [19]. Fabric and its evolution were incorporated in the dilatancy equation by Wan and Guo [56,57]. In this paper, the effect of initial anisotropy and its evolution is incorporated and verified with the experimental tests.

Oda et al. [35] conducted some triaxial tests on fine Toyoura sand ($D_{50} = 0.18$ mm, $c_u = 1.5$). Maximum and minimum void ratios were 0.99 and 0.63, respectively. They used strong particles to ensure minimal particle crushing. The specimens were sunk into a bucket filled with water and inclined at a tilting angle δ (Fig. 2). The sand was tapped sufficiently to provide specimens having a void ratio of about 0.67 to 0.68. Specimens were sheared in different tilting angles $\delta = 0°, 30°, 60°$ and $90°$ in triaxial compression tests. The microscopic examination in the vertical sections show that the apparent long axes of the particles were aligned parallel to the bedding angle, as shown in Fig. 3.

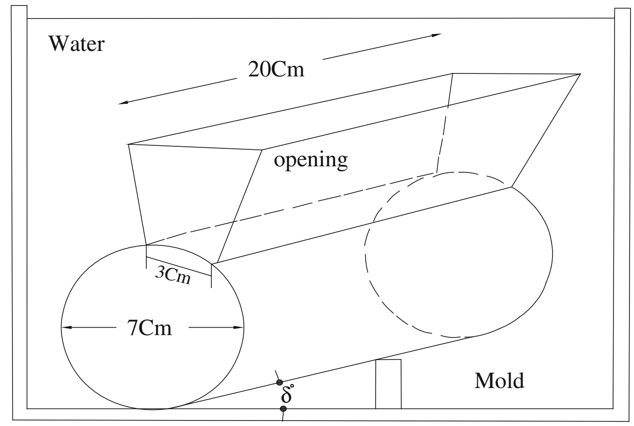

Fig. 2 Bucket used for deposition of the sands with a tilting angle δ (after Oda et al. [35])

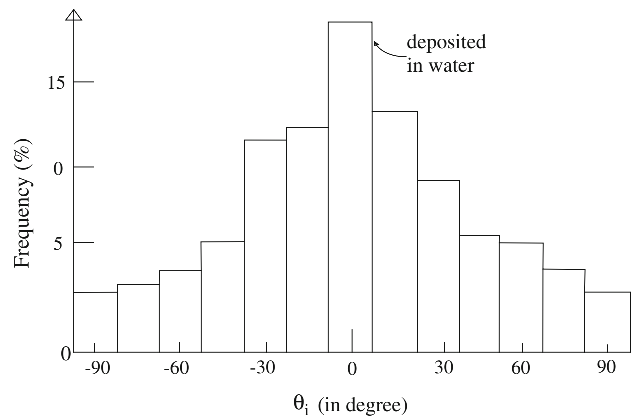

Fig. 3 Initial distribution of the contact normals for all samples with respect to their deposition angles

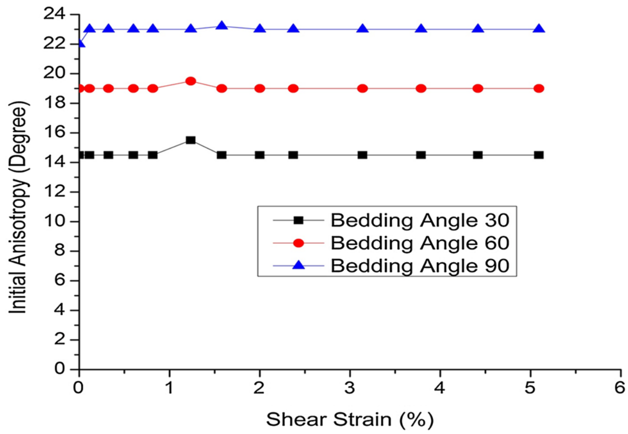

Fig. 4 Variation of initial anisotropy in the shearing process for $p_\circ = 0.5\,\text{kg/cm}^2$ and different bedding angles

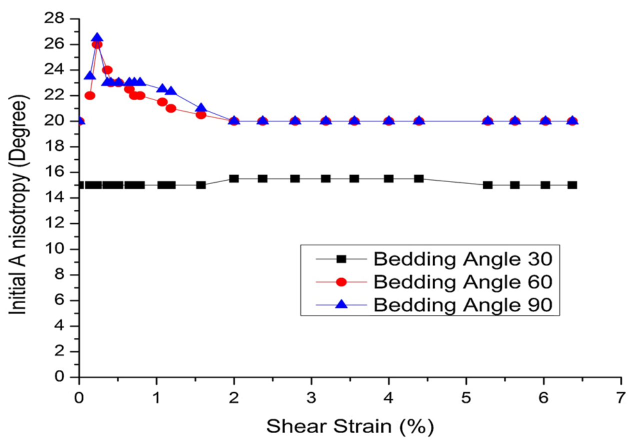

Fig. 5 Variation of initial anisotropy in the shearing process for $p_\circ = 2.0\,\text{kg/cm}^2$ with different bedding angles

Table 1 Constantans that are used for the simulations

M	1.25
c	0.008
x $p_\circ = 0.5\,\text{kg/cm}^2$	0.952
x $p_\circ = 2.0\,\text{kg/cm}^2$	0.80
n^d	0

Table 2 Constants used for the evolution of fabric

Non-coaxiality	z	λ
$\theta_f - \theta_\sigma > 60$	6.29	10
$33 < \theta_f - \theta_\sigma < 60$	4.2	10
$30 < \theta_f - \theta_\sigma < 33$	3.8	10
$23 < \theta_f - \theta_\sigma < 30$	4.2	10
$21 < \theta_f - \theta_\sigma < 23$	5.7	10
$13 < \theta_f - \theta_\sigma < 21$	2.3	10
$11 < \theta_f - \theta_\sigma < 13$	17	10
$\theta_f - \theta_\sigma < 11$	3	10

Assessment of constants in shearing process and their evolution is an important part of the simulation. The constants for modeling the dilatancy, fabric and its evolution can be divided into two categories, The first category includes z and λ; which depend on the non-coaxiality between stress and fabric. The other category includes $\cos 2(\beta_i - \beta_0)$, c and x. These are obtained from shear strength data set (see "Appendix"). The effect of inherent (or initial) anisotropy and induced anisotropy were included in the dilatancy formulation via $\cos 2(\beta_i - \beta_\circ)$, α and θ_f, respectively. The constants for inherent parameter have already been obtained from the shear strength data set. In order to model fabric and its evolution $\phi_{\mu mob}$ has a dominant effect. In Eq. (41) $\dot{\varepsilon}_v$ or dilatancy is not a dominant factor for calculation of $\phi_{\mu mob}$, therefor it is ignored for the first round of calculation. In the shearing process the contact normals change their direction with respect to maximum compression. For different bedding angles, the contact normals have different non-coaxiality (or deviation between stress and fabric), the non-coaxiality decreases with increasing of shear strain. For all samples with different bedding angles in the shearing process their contact normals changes but the constants depend on the magnitude of non-coaxiality; i.e. a sample could have varying magnitude of non-coaxiality from the beginning to failure. In spite of different bedding angles at the beginning of shearing, all samples have a certain magnitude of the constants at a specific range of non-coaxiality. The classical constants and the additional constants for non-coaxiality are the same for simulation. For this reason we used one set of constant for all samples with different bedding angles and different confining pressures during shearing process.

The parameters α and θ_f are obtained from Eqs. (39) and (38) respectively. The specimens with different tilting angles were sheared at the cell pressures of 0.5 and $2.0\,\text{kg/cm}^2$. The variation of the initial anisotropy (or initial real deviation angle) in the process of shearing for the different tilting angle and cell pressures are shown in Figs. 4 and 5. The magnitude of $\cos 2(\beta_i - \beta_\circ)$ is obtained by back calculation. The constants used to model the dilatancy are M, n^d, $\cos 2(\beta_i - \beta_\circ)$, z, and x that are presented in the Tables 1 and 2. The constant M is a classic parameter and easily obtained from experimental data [or from literature for Toyoura sand, (e.g., [20,21])]. Here, the effect of confining pressure was included via the parameter x, hence in this simulation the parameter n^d has been neglected.

In Figs. 6 and 7 the dilatancy obtained by Eq. (32) were compared with the experimental tests by Oda et al. [36]. The effects of inherent and induced anisotropy have been

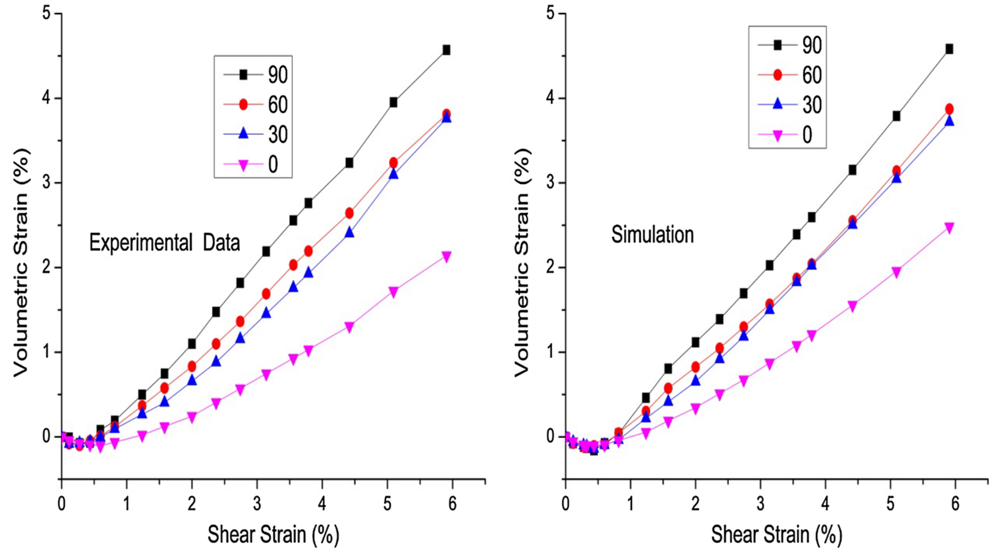

Fig. 6 Comparison between experimental data and the proposed dilatancy Eq. (32) for the confining pressure 0.5 kg/cm^2

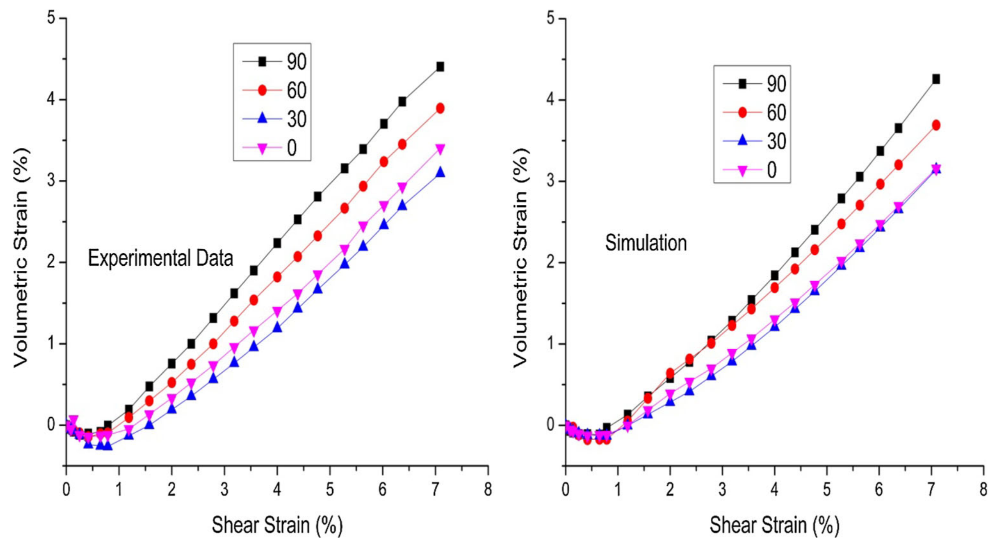

Fig. 7 Comparison between experimental data and the proposed dilatancy Eq. (32) for the confining pressure 2.0 kg/cm^2

included in these simulations. By increasing the tilting angle δ, the magnitude of the parameter α increases with increasing plastic shear deformation, and at the same time, the magnitude of θ_f decreases. The difference is due to the variation of the anisotropy parameters (α, θ_f and $\cos 2(\beta_i - \beta_\circ)$), since at the beginning of shearing the variation of the void ratio in different samples is not a dominant factor. The variation of the function $F(F_{ij}, \varepsilon_q^p)$ in the denominator of Eq. (32) is located in the range of the variation of N presented by Jefferies [13]. It is obvious that applying these equations with one set of constants can sufficiently model dilatancy in granular material.

8 Conclusion

Using micro-level analysis and the principles of thermodynamics a comprehensive flow rule has been proposed in which the effect of inherent and induced anisotropy is included. The internal work done by the internal forces and their counterparts strain has been related to the actual applied external loads. The dissipation mechanism in the granular materials was related to the macro-level dissipation mechanism. The applied and dissipation functions defining the contact normals distribution and the internal forces have been adapted from Rothenburg and Bathurst [47] and Radjai and Azema [42]. The variation of the contact normals or induced anisotropy was related to the variation of the degree of anisotropy α and the direction of the deposition angle of the particles mobilized inside the media, $\cos 2\beta_i$. The parameter $\cos 2(\beta_i - \beta_\circ)$ was applied to show the symbolic limited variation of the initial anisotropy. It was shown that fabric plays a "bridge-like" role in the non-coaxial flow rule. Non-coaxialties between stress-strain-fabric were attributed to the non-coaxiality between stress-fabric and strain-fabric. Unlike many other flow rules, in this new formulation the constants are depend on the non-coaxiality between stress and fabric. Verification of the formulation was carried out by simulation of experimental tests conducted by Oda et al. [36].

Appendix

Constants for fabric evolution

Assuming a symmetric second-rank tensor for the distribution of the contact normals (peanut shaped function $E(\theta) = (1/2\pi)(1+\alpha\cos 2(\theta-\theta_f)))$ and by focusing on two particle across a potential sliding plane, the following equations are obtained:

$$\alpha = \frac{(\tau/p)\cos\phi_{\mu mob} - \sin\phi_{\mu mob}}{\sin(2\theta_f + \phi_{\mu mob}) - ((\tau/p)\cos(2\theta_f + \phi_{\mu mob}))} \tag{45}$$

$$\tan^{-1}\left(\frac{\tau}{p}\right) = \frac{\theta_\sigma - \theta_f}{z} + \lambda\left(\frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_q}\right) + \phi_{\mu mob} \tag{46}$$

where z and λ are constants and irrespective to their bedding angles or confining pressures their magnitude depend only on the non-coaxiality between stress and fabric.

Constants for shear strength

Shaverdi et al. [50,51] proposed a new formulation for modeling of the yield surface (extended form of Mohr–Coulomb yield surface) in the triaxial case as follows:

$$f = q - \eta_y^f p_0 \tag{47}$$

and

$$\eta_y^f = \frac{\varepsilon_q^p}{c + \varepsilon_q^p}(1 + (1/2)\alpha\cos 2(\theta_f - \theta_\sigma)) \times \cos 2(\beta_i - \beta_\circ) M \exp(-n_b\psi) \tag{48}$$

c and $\cos 2(\beta_i - \beta_0)$ are the material constants. The constant $\cos 2(\beta_i - \beta_0)$ may readily be obtained by back calculation but as a rough estimation its value is close to the magnitude of the bedding angle $\cos\delta$ (for bedding angle δ between 15° to 45°). c is a constant that is used for modeling parabolic shape of the stress-strain behavior of soil. It is a soil constant which essentially scales the plastic strain since Eqs. (47, 48) are functions of ε_q^p/c.

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